Workout Exam Jan. 27, 2011

- **Exercise 1**

Relevant theory on sheet 15 chulp1

```
In[39]:= x5 = 1.37530
         x6 = 1.36593
         x7 = 1.37009
         dx7 = x7 - x6
         dx6 = x6 - x5
         lambda = dx7 / dx6

Out[39]= 1.3753

Out[40]= 1.36593

Out[41]= 1.37009

Out[42]= 0.00416

Out[43]= -0.00937
```

a

```
Out[44]= -0.44397
```

De uitdrukking schat p−x_n, dus x_n−p heeft net ander teken

```
In[45]:=
         aitken = lambda / (1 - lambda) * dx7
         better = x7 + aitken
```

b

```
Out[45]= -0.00127905
```

c

```
Out[46]= 1.36881
```

- **Exercise 2**

```
In[47]:= f = 4 * x^2 + y^2 - 4; g = x + y - Sin[x - y];
```

Chulp 2d

a sheet 6

b

c sheet 6

d sheet 6 + 3

```
In[48]:= fx = D[f, x]
         fy = D[f, y]
         gx = D[g, x]
         gy = D[g, y]

Out[48]= 8 x

Out[49]= 2 y

Out[50]= 1 - Cos[x - y]

Out[51]= 1 + Cos[x - y]

In[52]:= Jacf = {{fx, fy}, {gx, gy}}
         MatrixForm[Jacf]

Out[52]= {{8 x, 2 y}, {1 - Cos[x - y], 1 + Cos[x - y]}}
```

Out[53]//MatrixForm=

a $\begin{pmatrix} 8x & 2y \\ 1-\mathrm{Cos}[x-y] & 1+\mathrm{Cos}[x-y] \end{pmatrix}$

b $f_1(1,0) = 0$

$f_2(1,0) = 1 - \sin(1) = 1 - \left(1 - \frac{1}{6} + \right) \approx \frac{1}{6}$ klein

```
In[66]:= Jf = Jacf /. {x -> 1, y -> 0};
         MatrixForm[Jf]
         MatrixForm[N[Jf]]
         A = -Inverse[Jf];
```

Out[67]//MatrixForm=

$\begin{pmatrix} 8 & 0 \\ 1-\mathrm{Cos}[1] & 1+\mathrm{Cos}[1] \end{pmatrix}$

Out[68]//MatrixForm=

$J_f = \begin{pmatrix} 8. & 0. \\ 0.459698 & 1.5403 \end{pmatrix}$

$\frac{\partial g}{\partial x}\left(\boxed{\begin{smallmatrix}1\\0\end{smallmatrix}}\right) = 0$

$\longrightarrow I + A J_f = 0$

```
In[57]:= JacG = Simplify[IdentityMatrix[2] + A.Jacf]
         Simplify[JacG /. {x → 1, y → 0}]
         JGp = JacG /. {x -> 0.9986, y -> -0.1055}
         eiv = Eigenvalues[JGp]
         Abs[eiv]
         Norm[JGp, Infinity]
```

Out[57]= $\left\{\left\{1 - x, -\frac{y}{4}\right\}, \left\{\frac{-1 + x - x\,\mathrm{Cos}[1] + \mathrm{Cos}[x - y]}{1 + \mathrm{Cos}[1]}, \frac{y + 4\,\mathrm{Cos}[1] - y\,\mathrm{Cos}[1] - 4\,\mathrm{Cos}[x - y]}{4\,(1 + \mathrm{Cos}[1])}\right\}\right\}$

Out[58]= {{0, 0}, {0, 0}}

Out[59]= {{0.0014, 0.026375}, {-0.0590842, 0.0507948}}

Out[60]= {0.0260974 + 0.0307958 i, 0.0260974 - 0.0307958 i}

Out[61]= {0.0403665, 0.0403665} convergentie wordt bepaald door $\rho(JGp) = 0.0403668 << 1$

$\|JGp\|_\infty$ = 0.109879 $\rho(A) \leq \|A\|_\infty \rightarrow$ convergentie

Why

$$x_{n+1} = g(x_n)$$
$$p = g(p)$$
$$x_{n+1} - p = \underbrace{\frac{\partial g}{\partial x}(p)}\,(x_n - p) + O(\|x_n - p\|^2)$$

$\rho < 1$ geeft convergentie.

Answers Exam January 27, 2011

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January 27, 2011

Exerc 3

a. LN interpFW 1, book

$$p_2(x) = \frac{(x-x_1)(x-x_2)}{(x_0-x_1)(x_0-x_2)}y_0 + \frac{(x-x_0)(x-x_2)}{(x_1-x_0)(x_1-x_2)}y_1 + \frac{(x-x_0)(x-x_1)}{(x_2-x_0)(x_2-x_1)}y_2$$

b. LN interpFW 10

$$p_2(x) = f(x_0) + (x - x_0)f[x_0, x_1] + (x - x_0)(x - x_1)f[x_0, x_1, x_2]$$

c. LN interpFW 12. Unlike the form in a, the form in b can be written in a nested way.

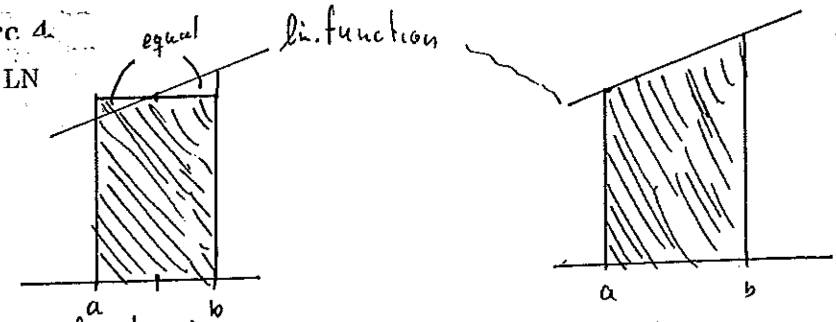
$$p_2(x) = f(x_0) + (x - x_0)\{f[x_0, x_1] + (x - x_1)f[x_0, x_1, x_2]\}$$

So once all the divided differences are computed, which is in general $O(n^2)$ work for $O(n)$ interpolation points, it only costs n additions and multiplications to evaluate the interpolating polynomial

Exerc 4

chintgr FW sheets: 5, 9, 10 sketches made during lectures

a. LN



Area of rectangle and trapezium are the same

b. We have $I = I(h) + ch^3 + O(h^5)$ and $I = I(2h) + c8h^3 + O(h^5)$. We have to get rid of the $O(h^3)$ error term, which occurs by multiply the first by 8 and subtract the second one. This gives $7I = 8I(h) - I(2h) + O(h^5)$. Dividing by 7 we obtain

sheet 26 and many others

$$I = \frac{8I(h) - I(2h)}{7} + O(h^5)$$

So an $O(h^5)$ approximation to I is given by $(8I(h) - I(2h))/7$.

Exerc 5

chgdv FW

sheets 21-23 + 41

a. The trapezium method is given by $w_{n+1} = w_n + \Delta t \frac{1}{2}(f(t_n, w_n) + f(t_{n+1}, w_{n+1}))$, hence we find

$$\begin{bmatrix} w_1^{(n+1)} \\ w_2^{(n+1)} \end{bmatrix} = \begin{bmatrix} w_1^{(n)} \\ w_2^{(n)} \end{bmatrix} + \Delta t \frac{1}{2}\left(\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} w_1^{(n)} + w_1^{(n+1)} \\ w_2^{(n)} + w_2^{(n+1)} \end{bmatrix} + 2 \begin{bmatrix} 1 \\ -2 \end{bmatrix} \right)$$

with

$$\begin{bmatrix} w_1^{(0)} \\ w_2^{(0)} \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

sheets 11, 39, 40, 50

b. Apply the methods to the test equation $y' = \lambda y$. Forward Euler $w_{n+1} = (1 + \Delta t \lambda) w_n$. The region of absolute stability is that part of the complex plane where $|1 + z| < 1$, hence z replaces $\lambda \Delta t$. Similarly one finds for backward Euler method $|1 - z| > 1$ and for the trapezium method $|2 + z| < |2 - z|$. The respective regions of absolute stability are interior of circle with radius 1 and center -1, exterior of circle with radius 1 and center +1, the left half plane.

sheets 21-23

c. We have to look to the eigenvalues of the matrix which are $\pm i$. Δt should be such that $\Delta t \lambda$ is in the domain. For Forward Euler there is no such Δt for backward Euler any Δt can be taken. For the trapezium method we are on the boundary of the domain. So there is no amplification nor damping of the error. This is acceptable in this case where the solution also does not have damping nor amplification. So also for the trapezium method we may take any time step Δt.

Exercise 6

chla 2

sheet 3

a. For Jacobi method we have the splitting $A = D + B$ where D is the diagonal.

$$\begin{bmatrix} 3 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix} x_{n+1} = - \begin{bmatrix} 0 & -1 & -1 \\ -1 & 0 & 0 \\ 0 & -1 & 0 \end{bmatrix} x_n + \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

or

$$x_{n+1} = - \underbrace{\begin{bmatrix} 0 & -1/3 & -1/3 \\ -1/2 & 0 & 0 \\ 0 & -1/2 & 0 \end{bmatrix}} x_n + \begin{bmatrix} 1/3 \\ 1/2 \\ 1/2 \end{bmatrix}$$

sheet 8, 9

b. The matrix in the rhs is also the iteration matrix. The iteration converges if the eigenvalues of this matrix are all less than one in magnitude. One can compute eigenvalues which may be rather nasty here, but the spectral radius is also bounded by the infinity norm of the matrix, which is clearly 2/3. Hence the method converges.

sheet 10

c. For the Gauss Seidel method we find the iteration

$$\begin{bmatrix} 3 & 0 & 0 \\ -1 & 2 & 0 \\ 0 & -1 & 2 \end{bmatrix} x_{n+1} = - \begin{bmatrix} 0 & -1 & -1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} x_n + \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

Exercise 7

a. LN chpdv: Use

sheet 2

$$u(x_{i\pm 1}) = u(x_i) \pm hu'(x_i) + \frac{h^2}{2}u''(x_i) \pm \frac{h^3}{6}u'''(x_i) + \frac{h^4}{24}u''''(x_i)....$$

Add the expression with the + to the expression with the - to get rid of the first derivative term. Doing this also the third derivative cancels out. We get

$$u(x_{i+1}) + u(x_{i-1}) = 2u(x_i) + h^2u''(x_i) + \frac{h^4}{12}u''''(x_i)....$$

After rewriting one obtains the desired result

$$u_{xx}(x_i) = \frac{u(x_{i+1}) - 2u(x_i) + u(x_{i-1})}{\Delta x^2} - \frac{h^2}{12}u''''(x_i) + ...$$

The second argument t does not play a role here.

b. LN chpdv:1-4 book 11.3, 12.2.

sheet 3

$$\frac{dv_i}{dt}(t) = \kappa\frac{v_{i+1}(t) - 2v_i(t) + v_{i-1}(t)}{\Delta x^2} \text{ for } i = 1, ..., m-1 \qquad (1)$$

where $v_i(t)$ approximates $u(x_i, t)$. The initial condition is $v_i(0) = \sin(\pi i\Delta x)$ and the boundary conditions $v_0(t) = 0$ and $v_m(t) = \sin^2(\pi t)$. This completes the discretization in space.

c. Next we apply the explicit Euler method to this system of equations. Let $w_i^n \approx v_i(n\Delta t)$. Then we write

sheet 4

$$w_i^{n+1} = w_i^n + \Delta t\kappa\frac{w_{i+1}^n - 2w_i^n + w_{i-1}^n}{\Delta x^2}$$

with initial condition $w_i^0 = \sin(\pi i\Delta x)$ and boundary conditions $w_0^n = 0$, $w_m^n = \sin^2(\pi n\Delta t)$.

d. LN chpdv: 7-9. book 12.3
For the forward Euler method there is a time step restriction, for the backward Euler there is not.